Spring 2010 Math 245-2 Exam 2 Solutions

One quarter of students scored 58-68, one quarter scored 68-73, one quarter scored 73-78, one quarter scored 78-94. In particular, the median was 73, the low was 58, and the high was 94.

Problem 1. Carefully define each of the following terms:

a. constructive proof
An existential proof is constructive if it explicitly produces the desired object (or gives an algorithm to produce it).

b. floor
The floor of real number x is the largest integer n with $n \leq x$.

c. odd
A number x is odd if there is an integer n with $x = 2n + 1$.

d. irreducible
A number is irreducible if it cannot be factored into two nonunits.

Problem 2. Prove that $\sqrt{5}$ is irrational.
Argue by way of contradiction. If $\sqrt{5}$ were rational, there would be integers p, q with $\sqrt{5} = \frac{p}{q}$. We may assume without loss of generality that p, q have no common factors. Squaring both sides and multiplying by q^2 we get $5q^2 = p^2$. Hence $5|p \cdot p$. Since 5 is prime, $5|p$ or $5|p$, so either way $5|p$. Hence there is some integer s with $5s = p$; we plug in to get $5q^2 = p^2 = (5s)^2 = 5^2s^2$. Cancelling, we get $q^2 = 5s^2$, so $5|q \cdot q$. Since 5 is prime, $5|q$ as earlier. Hence 5 is a common factor of p, q, a contradiction.

Problem 3. Use the Euclidean algorithm to first find $\gcd(21, 15)$, then to express $\gcd(21, 15)$ as a linear combination of 15 and 21.
$21 = 1 \cdot 15 + 6,\ 15 = 2 \cdot 6 + 3,\ 6 = 2 \cdot 3 + 0$. Hence $\gcd(21, 15) = 3$. Working backwards, $3 = 15 - 2 \cdot 6 = 15 - 2(21 - 1 \cdot 15) = 3 \cdot 15 - 2 \cdot 21$.

Problem 4. Prove that, in the reals, the product of an irrational and a nonzero rational is irrational.
Let x be irrational and y be a nonzero rational. Since y is rational there are integers m, n with $n \neq 0$ where $y = \frac{m}{n}$. Note that $m \neq 0$ since $y \neq 0$. Argue by way of contradiction; we assume that xy is in fact rational. Hence there are integers c, d with $d \neq 0$ where $xy = \frac{c}{d}$. But now $x\frac{m}{n} = \frac{c}{d}$, so $x = \frac{cn}{dm}$. This expresses x as a ratio of two integers, and $dm \neq 0$. Hence x is rational, but this is a contradiction.

Problem 5. Prove or disprove that, $\forall x \in \mathbb{R}$, $\lfloor -x \rfloor = -\lfloor x \rfloor$.
This is false. Take $\mathbf{x = 0.5}$ (many other counterexamples are possible). $\lfloor -\mathbf{x} \rfloor = -\mathbf{1}$, while $-\lfloor \mathbf{x} \rfloor = \mathbf{0}$.

Problem 6. Prove or disprove that, $\forall x \in \mathbb{R}$, $\lfloor -x \rfloor = -\lceil x \rceil$.
SOLUTION 1: Proof by cases. If x is an integer, both sides equal $-x$. If $x = n+y$, where $0 < y < 1$ and n is an integer, then $\lfloor -x \rfloor = \lfloor -n-y \rfloor = -n + \lfloor -y \rfloor = -n-1$. On the other hand, $-\lceil x \rceil = -\lceil n+y \rceil = -n - \lceil y \rceil = -n-1$.

SOLUTION 2: Set $a = \lfloor -x \rfloor, b = -\lceil x \rceil$. By the definition of floor, $-x-1 < a \leq -x$. By the definition of ceiling, $x \leq \lceil x \rceil < x+1$, hence $-x \geq -\lceil x \rceil = b > -x-1$. Hence both a, b are integers in the interval $(-x-1, -x]$; but this interval has only one integer so $a = b$.

Problem 7. Prove or disprove that, $\forall x \in \mathbb{R}$, $-|x| \leq x \leq |x|$.
Proof by cases. If $x \geq 0$, then $|x| = x$. Certainly then $|x| = x \geq x$, and also $-|x| \leq 0 \leq x$, so $-|x| \leq x \leq |x|$. On the other hand, if $x < 0$, then $|x| = -x$. Then $-|x| = x \leq x$, and also $|x| > 0 > x$ so $-|x| \leq x \leq |x|$.

Problem 8. Consider the sequence given by $a_1 = 1, a_{n+1} = 3a_n + 3^n$ (for $n \geq 1$). Prove that $a_n = n3^{n-1}$.
Let $S(n)$ denote the proposition $a_n = n3^{n-1}$. Proof by induction on n. Base case, $n = 1$, $a_1 = 1 = 1 \cdot 3^0$. Now, assume $S(n)$, $a_n = n3^{n-1}$. By the recurrence, $a_{n+1} = 3a_n + 3^n = 3(n3^{n-1}) + 3^n = n3^n + 3^n = (n+1)3^n$. This is $S(n+1)$, hence we've proved $S(n) \to S(n+1)$.

Problem 9. Prove that $\left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} 3^n & n3^{n-1} \\ 0 & 3^n \end{smallmatrix}\right)$.
Let $S(n)$ denote the proposition $\left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} 3^n & n3^{n-1} \\ 0 & 3^n \end{smallmatrix}\right)$. Proof by induction on n. Base case, $n = 1$, $\left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)^1 = \left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3^1 & 1 \cdot 3^0 \\ 0 & 3^1 \end{smallmatrix}\right)$. We now assume $S(n)$; that is, $\left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} 3^n & n3^{n-1} \\ 0 & 3^n \end{smallmatrix}\right)$. Now, $\left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)^{n+1} = \left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3^n & n3^{n-1} \\ 0 & 3^n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3^{n+1} & 3n3^{n-1}+3^n \\ 0 & 3^{n+1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3^{n+1} & (n+1)3^n \\ 0 & 3^{n+1} \end{smallmatrix}\right)$. This proves $S(n+1)$.

Problem 10. For an arbitrary set of numbers S, recall that x is a *unit* if there is some y such that $xy = 1$. Prove that the product of two units is a unit.
Suppose that x, x' are any two units. We want to prove that their product xx' is also a unit. Because x, x' are units, there are numbers y, y' where $xy = 1 = x'y'$. Now, set $z = y'y$, a number. We have $(xx')z = xx'y'y = x(x'y')y = x(1)y = xy = 1$. Hence there is a number (namely z) such that $(xx')z = 1$, so xx' is a unit.